PERTH MODERN SCHOOL
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Perth Modern School

Semester Two Examination, 2016

Question/Answer Booklet

MATHEMATICS SPECIALIST UNITS 3 AND 4

Section Two: Calculator-assumed

Student Name: SOL EWSHONS

Teachers Name: ______________________

Student Number: In figures | | | | | | | | |

In words ______________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time for section: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in the WACE examinations

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	97	65
			Total	149	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Booklet.

Section Two: Calculator-assumed **65% (97 Marks)**

This section has **twelve (12)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 100 minutes.

Question 9 **(17 marks)**

Consider the functions $f(x) = ln(x)$ and $g(x) = \sqrt{x}$ for $x > 0$.

(a) (i) Determine the expression for $y = f(g(x))$ and explain why the function is defined.

$y = f(g(x)) = f(\sqrt{x}) = ln\sqrt{x} = \frac{1}{2}ln(x)$ ✓

$ln(x)$ is defined for $x > 0$ and this is the given domain so

$y = f(g(x))$ is defined. ✓ (2)

(ii) Explain why the function $y = f(g(x))$ has an inverse.

$y = f(g(x)) = \frac{1}{2}ln(x)$ is monotonically increasing, ✓

so $y = f(g(x))$ has an inverse. (1)

(iii) Find the equation of the inverse of the function $y = f(g(x))$.

$y = \frac{1}{2}ln(x)$

To obtain the inverse, switch x and y

$x = \frac{1}{2}ln(y) \Rightarrow 2x = ln\,y$ ✓ (3)

$\therefore\ y = e^{2x}$ or if $f(g(x)) = f \circ g$ then $(f \circ g)^{-1}(x) = e^{2x}$

✓

(b) (i) Find the domain such that the function $y = g(f(x))$ exists. (3)

$g(f(x)) = g(ln(x)) = \sqrt{ln(x)}$ ✓

$\sqrt{ln(x)}$ is defined for $ln(x) \geq 0$ ✓

$ln(1) = 0$

$\therefore\ g(f(x))$ is defined for $x \geq 1$ ✓

(ii) Determine whether the function $y = g(h(x))$ is a one to one function given $h(x) = e^{-x^2}$.

$y = g(h(x)) = g\left(e^{-x^2}\right) = \sqrt{e^{-x^2}}$ ✓

$g(h(1)) = \sqrt{e^{-(1)^2}} = \sqrt{e^{-1}} = \dfrac{1}{\sqrt{e}}$

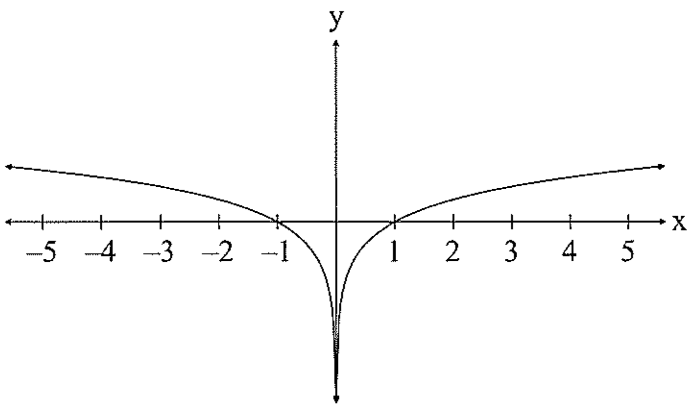
$g(h(-1)) = \sqrt{e^{-(-1)^2}} = \sqrt{e^{-1}} = \dfrac{1}{\sqrt{e}}$ ✓

$\therefore g(h(1)) = g(h(-1))$

The function $y = g(h(x))$ is not a 1-1 function. ✓ (3)

(c) (i) Given $p(x) = |x|$ sketch $y = f(p(x))$ on the set of axes below. (3)

$y = f(p(x)) = f(|x|) = ln|x|$ ✓



✓✓

(ii) Find $f\left(p\left(-e^{-3}\right)\right)$

$f\left(p\left(-e^{-3}\right)\right) = f\left(\left|-e^{-3}\right|\right) = f\left(e^{-3}\right) = ln\left(e^{-3}\right) = -3\,ln\,e = -3$ (2)

✓ ✓

Question 10 **(5 marks)**

Use the expansion of $\left(cos(x)+i\,sin(x)\right)^3$ to show that $cos(3x)=4cos^3(x)-3cos(x)$.

$$\left(cos(x)+i\,sin(x)\right)^3 = cos(3x)+i\,sin(3x) \quad \checkmark$$

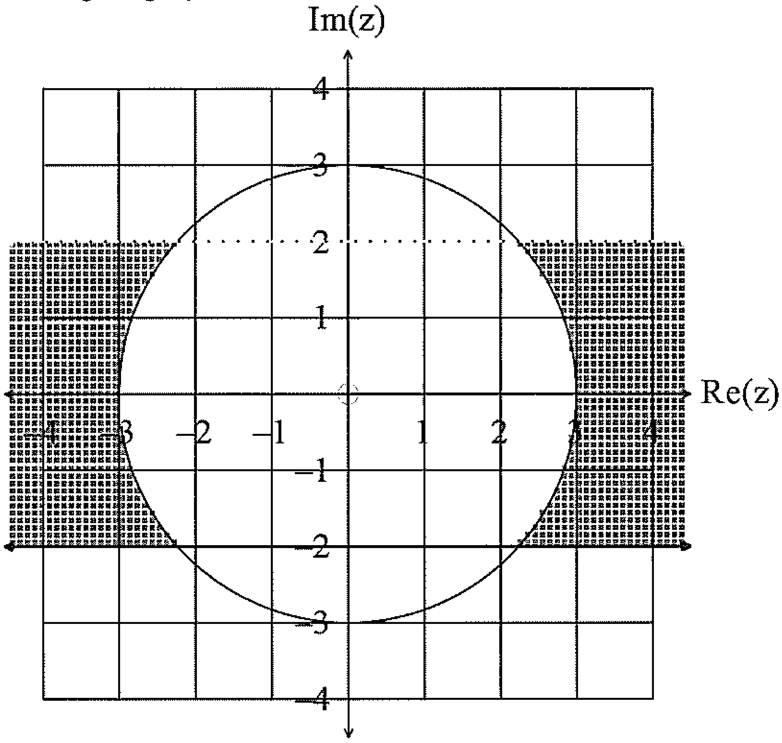
$$LHS\ \left(cos(x)+i\,sin(x)\right)^3 = cos^3(x)+3i\,cos^2(x)\,sin(x)-3\,cos(x)\,sin^2(x)\overset{\checkmark}{}-i\,sin^3(x) \quad \checkmark$$

Equating the reals:

$$\begin{aligned} cos(3x) &= cos^3(x)-3\,cos(x)\overset{\checkmark}{}sin^2(x) \\ &= cos^3(x)-3\,cos(x)\left(1-cos^2(x)\right) \quad \checkmark \\ &= cos^3(x)-3\,cos(x)+3\,cos^3(x) \\ cos(3x) &= 4cos^3(x)-3\,cos(x) \end{aligned}$$

Question 11 **(10 marks)**

(a) Define the shaded region graphed on the set of axes below. (3)



(a) $\{z: 3 \le |z| \cap -2 \le Im(z) < 2\}$
✓ ✓ ✓ (3)

(b) Given $z_1 = a + bi$ and $z_2 = c + di$ prove that $|z_1| \times |z_2| = |z_1 \times z_2|$.

$$|z_1 \times z_2| = |(a+bi) \times (c+di)|$$
$$= |ac - bd + i(ad + bc)|$$
$$= \sqrt{(ac-bd)^2 + (ad+bc)^2}$$
$$= \sqrt{a^2c^2 - \cancel{2abcd} + b^2d^2 + a^2d^2 + \cancel{2abcd} + b^2c^2} \quad ✓$$
$$= \sqrt{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2}$$
$$= \sqrt{a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2}$$
$$= \sqrt{a^2(c^2+d^2) + b^2(c^2+d^2)} \quad ✓$$
$$= \sqrt{(a^2+b^2)(c^2+d^2)}$$
$$= \sqrt{a^2+b^2} \times \sqrt{c^2+d^2} \quad ✓$$
$$= |z_1| \times |z_2|$$
$$\therefore \ |z_1 \times z_2| = |z_1| \times |z_2|$$
(3)

(c) Sketch $|z-3+3i|=2|z|$ on the set of axes below.
Hint: Put $z=x+iy$ (4)

$|z-3+3i|=2|z|$

$|(x-3)+(y+3)i|=2|x+iy|$

$\sqrt{(x-3)^2+(y+3)^2}=2\sqrt{x^2+y^2}$

$x^2-6x+9+y^2+6y+9=4x^2+4y^2$ ✓

$3x^2+3y^2+6x-6y-18=0$

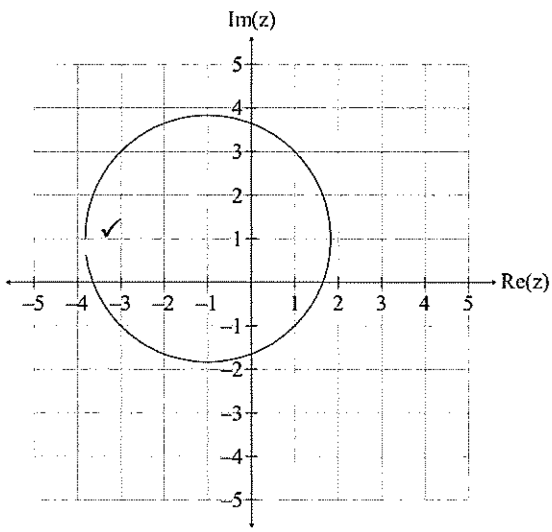
$x^2+y^2+2x-2y-6=0$ ✓

$(x^2+2x+1)+(y^2-2y+1)=6+1+1$

$(x+1)^2+(y-1)^2=8$

Circle with $C(-1,1)$ $\quad r=\sqrt{1+1-(-6)}$

✓ $\quad r=\sqrt{8}$



(4)

Question 12 **(6 marks)**

The line $r = (2,{}^{-}1,3) + \lambda(1,2,0)$ and the plane $\boldsymbol{r} \cdot \langle \boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k} \rangle = 2$ intersect at point P.

(a) Determine the coordinates of P. (3)

$x = 2+\lambda \quad y = -1+2\lambda \quad z = 3$

Now $\langle 2+\lambda, -1+2\lambda, 3 \rangle \cdot \langle 1, -1, 2 \rangle = 2$

ie $2+\lambda + 1 - 2\lambda + 6 = 2$

ie $9 - \lambda = 2$

ie $\lambda = 7$

$\therefore$ P has co-ordinates $(9, 13, 3)$

(b) Determine the size of the angle between the line and the plane. (3)

$\langle 1, 2, 0 \rangle$ is direction of line

$\langle 1, -1, 2 \rangle$ is direction of normal

$\therefore \quad \langle 1, 2, 0 \rangle \cdot \langle 1, -1, 2 \rangle = \sqrt{5} \times \sqrt{6} \cos\theta$

ie $\cos\theta = -\frac{1}{\sqrt{30}}$

ie $\theta = 100.51°$ (Angle Between line & Normal to plane)

$\therefore$ $100.51 - 90 = 10.51°$ is angle between line & plane

Question 13 **(4 marks)**

Two magpies see an insect on a branch of a tree at T(3, -4,1) at the same instant. Magpie 1 is at A(-3,4,0) and magpie 2 is at B(-5,4,0). Both magpies immediately took off towards the insect with

velocities of $\begin{pmatrix} 3 \\ -4 \\ 0.5 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ -4 \\ 0.5 \end{pmatrix}$ ms^{-1} respectively.

Which magpie gets to the insect first? (4)

$$\boldsymbol{AT} = \begin{pmatrix} 6 \\ -8 \\ 1 \end{pmatrix} \quad r_1(t) = \begin{pmatrix} -3 \\ 4 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ -4 \\ 0.5 \end{pmatrix}$$

$$r_1(2) = \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix} \Rightarrow \text{Magpie}_1 \text{ takes 2 seconds} \quad \checkmark$$

$$\boldsymbol{BT} = \begin{pmatrix} 8 \\ -8 \\ 1 \end{pmatrix} \quad r_2(t) = \begin{pmatrix} -5 \\ 4 \\ 0 \end{pmatrix} + t\begin{pmatrix} 4 \\ -4 \\ 0.5 \end{pmatrix} \quad \checkmark$$

$$r_2(2) = \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix} \Rightarrow \text{Magpie}_2 \text{ takes 2 seconds} \quad \checkmark$$

Both magpies arrive at the point where the insect is at the same time.
(tug of war?)

Question 14 **(9 marks)**

The position of a small body relative to the origin at any time *t* seconds is given by

the vector function $\mathbf{r}(t) = 24sin\left(\frac{\pi t}{6}\right)\mathbf{i} + 24cos\left(\frac{\pi t}{6}\right)\mathbf{j},\ t \geq 0$.

(a) Determine an expression for $\mathbf{v}(t)$, the velocity of the body, and prove that the velocity of the small body is always perpendicular to its position vector for all *t*. (4)

if $\mathbf{r}(t) = 24\,sin\left(\frac{\pi t}{6}\right)\mathbf{i} + 24\,cos\left(\frac{\pi t}{6}\right)\mathbf{j}$

then $\mathbf{v}(t) = 4\pi\,cos\left(\frac{\pi t}{6}\right)\mathbf{i} - 4\pi\,sin\left(\frac{\pi t}{6}\right)$

now $\mathbf{v}(t) \bullet \mathbf{r}(t) = 0$

$\therefore \mathbf{v}(t) \perp \mathbf{r}(t)$

(b) Show that the body is travelling with constant speed and describe the motion of the body. (3)

$|\mathbf{v}| = \sqrt{16\pi^2\left(sin^2\left(\frac{\pi t}{6}\right) + cos^2\left(\frac{\pi t}{6}\right)\right)}$

ie $|\mathbf{v}| = 4\pi$, independent of t

The body is in circular motion.

(c) Explain why $\int_0^T |\mathbf{v}(t)|\,dt > \left|\int_0^T \mathbf{v}(t)\,dt\right|$ for any $T > 0$. (2)

$\int_0^T |\mathbf{v}(t)|\,dt$ represents the distance travelled on the curved path for the time interval 0 *to* T,

where as $\left|\int_0^T \mathbf{v}(t)\,dt\right|$ is the shortest distance between where it was at $t = 0$ to where it is at $t = T$.

Question 15 **(8 marks)**

The standard deviation of the lifetimes of Bright light bulbs is 770 hours.
Quality control experts plan to estimate μ, the mean lifetime of Bright bulbs, using the mean lifetime of a random sample of Bright bulbs.

(a) The experts would like to be 95% confident that the mean lifetime of bulbs in the sample is within 15 hours of μ. How large a sample should they take? (3)

$15 = 1.96\left(\frac{770}{\sqrt{n}}\right)$

in $n = 10\,123$

they should take a sample of at least 10 123.

(b) Suppose that a random sample of 100 Bright light bulbs is taken, and the mean lifetime of these bulbs is 10540 hours.
Based on this sample, determine a 90% confidence interval for μ. (3)

90% interval is $10540 \pm 1.645\left(\frac{770}{\sqrt{100}}\right)$

$= (10413.3,\ 10666.6)$.

(c) The manufacturer claims that the mean lifetime of Bright Bulbs is at least 10 000 hours. Does the sample in Part (b) provide a strong reason to doubt this claim? Justify your answer. (2)

Yes. 10 000 lies outside the 90% CI

Question 16

(3 marks)

A coffee machine is intended to produce cups of coffee with a mean temperature between 74 °C and 78 °C. However, the temperature of coffee produced is in fact uniformly distributed between 70 °C and 80 °C, with a mean of 75 °C and a standard deviation of 2.89 °C.

Use the Central Limit Theorem to estimate the probability that the mean temperature of the next 50 cups of coffee produced by the machine will lie between 74 °C and 78 °C.

Give the answer correct to **two (2)** decimal places.

Solution
Let X be the mean temperature of the 50 cups.
Then $X \sim N\left(75, \left(\frac{2.89}{\sqrt{50}}\right)^2\right)$
$P(74 < X < 78) = 0.99$
$\text{normCDf}\left(74, 78, \frac{2.89}{\sqrt{50}}, 75\right)$ 0.992792186
Specific behaviours
✓ identifies that a normal distribution is required ✓ uses correct parameters for the distribution ✓ calculates probability, correct to two decimal places

Question 17 **(8 marks)**

(a) Given $\frac{dy}{dx} = 2x(3y-2)$ find an expression for y in terms of x given that (1,1) belongs to the curve. (4)

$$\int \frac{dy}{(3y-2)} = \int 2x\,dx \quad ✓$$

$$\frac{ln|3y-2|}{3} = x^2 + c \quad ✓$$

$$(1,1) \quad \frac{ln 1}{3} = 1 + c \Rightarrow c = -1$$

$$\therefore \frac{ln(3y-2)}{3} = x^2 - 1 \quad ✓$$

$$(3y-2) = e^{3x^2-3}$$

$$y = \frac{e^{3x^2-3}+2}{3} \quad ✓$$

(4)

(b) If $a = 4 - x$ cm s^{-2} and when $x = 1$ cm, $v = 3$ cm s^{-1}, then find the velocity when $x = 2$ cm.

(4)

HINT: $\frac{v^2}{2} = \int a\,dx$

$\frac{v^2}{2} = \int a\,dx$

$\frac{v^2}{2} = \int (4 - x)\,dx$

$\frac{v^2}{2} = 4x - \frac{x^2}{2} + c$ ✓

$x = 1, v = 3$

$\frac{9}{2} = 4 - \frac{1}{2} + c \Rightarrow c = 1$

$\frac{v^2}{2} = 4x - \frac{x^2}{2} + 1$ ✓

$v^2 = 8x - x^2 + 2 \quad v > 0$ (when $x = 1, v = 3$) ✓

$\therefore \; v = \sqrt{8x - x^2 + 2}$

At $x = 2 \quad v = \sqrt{14}$ cm s^{-1} ✓

(4)

Question 18 **(6 marks)**

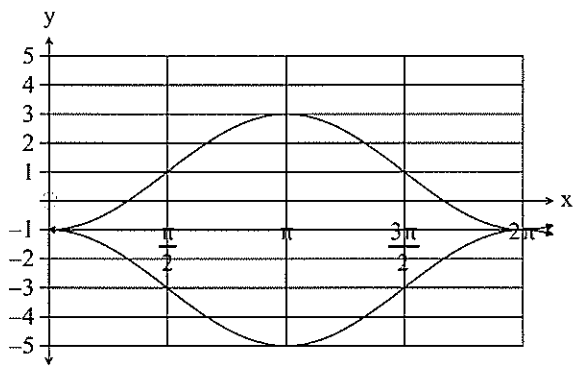
Given the functions $g(x) = 2\cos(x) - 3$ and $h(x) = 1 - 2\sin\left(x + \frac{\pi}{2}\right)$ defined on the interval $[0, 2\pi]$.

(a) find the exact x values where $g(x) = h(x)$. (3)

$x = 0,\ x = 2\pi$ ✓✓✓

(b) hence determine the area enclosed between the two curves correct to three decimal places. (3)

Area = $\int_0^{2\pi} \left(1 - 2\sin\left(x + \frac{\pi}{2}\right) - \left(2\cos(x) - 3\right)\right) dx = 25.13 \text{units}^2$ ✓✓✓



Question 19 **(12 marks)**

(a) The logistic differential equation is $\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$ where P represents population, K the carrying capacity and k is a positive constant.

(i) Explain what happens to the term $\left(1 - \frac{P}{K}\right)$ as $t \to \infty$ and how the change affects the growth curve.

As the population P gets closer to the carrying capacity K, then the factor $\left(1 - \frac{P}{K}\right)$ approaches 0, and $\frac{dP}{dt} \to 0$. This means the population is no ✓ longer increasing but becomes stable and the population approaches a number and remains close to that number. ✓ (2)

(ii) Show that $P = \frac{K}{1 + Ae^{-kt}}$ where $A = \frac{K - P_0}{P_0}$. (5)

HINT: Use partial fractions.

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right) = \frac{kP}{K}(K - P)$$

$$so \ \frac{dt}{dP} = \frac{K}{k}\left(\frac{1}{P(K-P)}\right)$$

$$\frac{k}{K}t = \int\left(\frac{1}{P(K-P)}\right)dP \quad ✓$$

Using partial fractions,

$$\frac{1}{P(K-P)} = \frac{a}{P} + \frac{b}{K-P}$$

$$= \frac{a(K-P) + bP}{P(K-P)}$$

$$\frac{0 \times P + 1}{P(K-P)} = \frac{P(b-a) + Ka}{P(K-P)}$$

$0 = b - a$ and $1 = Ka$

$a = b$ and $a = \frac{1}{K} = b$ ✓

$$\frac{1}{P(K-P)} = \frac{1}{KP} + \frac{1}{K(K-P)}$$

so $\frac{k}{K}t = \int\left(\frac{1}{P(K-P)}\right)dP$ becomes ✓

$\frac{k}{K}t = \int \frac{1}{KP} + \frac{1}{K(K-P)}dP$

$\frac{k}{K}t = \frac{1}{K}\left[\int \frac{1}{P}dP + \int \frac{1}{(K-P)}dP\right]$

$kt = \ln P + (-1)\ln(K-P) + C$

$kt - C = \ln\left(\frac{P}{K-P}\right)$

$\frac{P}{K-P} = e^{kt-C}$

$\frac{K-P}{P} = Ae^{-kt}$ $where\ A = e^{C}$ ✓

$At\ t = 0, P = P_0$

$\frac{K-P_0}{P_0} = Ae^0 = A$ so $A = \frac{K-P_0}{P_0}$

$\frac{K-P}{P} = Ae^{-kt}$

Rearrange the formula to get P ✓

$K - P = PAe^{-kt}$

$K = P(1 + Ae^{-kt})$

$\therefore\ P = \frac{K}{1+Ae^{-kt}}$ with $A = \frac{K-P_0}{P_0}$

(5)

(b) The <u>total number</u> of ebola cases (N) in Sierra Leone can be represented by the function $N = \frac{13809}{1+55.84e^{-0.67953t}}$ where t represents the number of months since February 2014.

(i) Determine the number of cases in February 2015 and in August 2015.

February 2015, $t = 12$ August 2015, $t = 18$

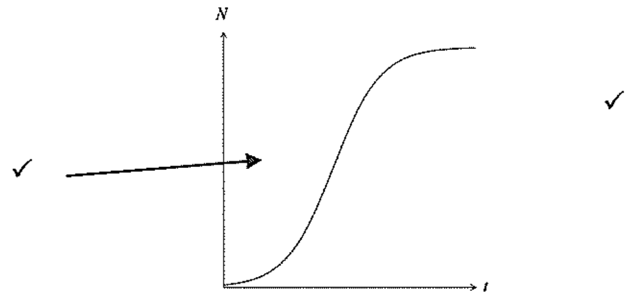
$N = \frac{13809}{1+55.84e^{-0.67953t}}$

$N(12) = 13\ 591$ ✓ $N(18) = 13\ 805$ ✓ (2)

(ii) Explain how to determine when the rate of the number of ebola cases stops increasing (but is still positive).
Refer to the graph below to indicate the point where this occurs.

The rate of growth of the number of patients with ebola stops increasing when

$\frac{d}{dt}\left(\frac{dN}{dt}\right)=0$ *i.e.* $\frac{d^2N}{dt^2}=0$. This is at the point of inflection.



(2)

(iii) The expected number of total cases

$$=\lim_{x\to\infty}\left(\frac{13809}{1+55.84e^{-0.67953t}}\right)$$
$$=13809. \quad ✓$$

(1)

Question 20 **(5 marks)**

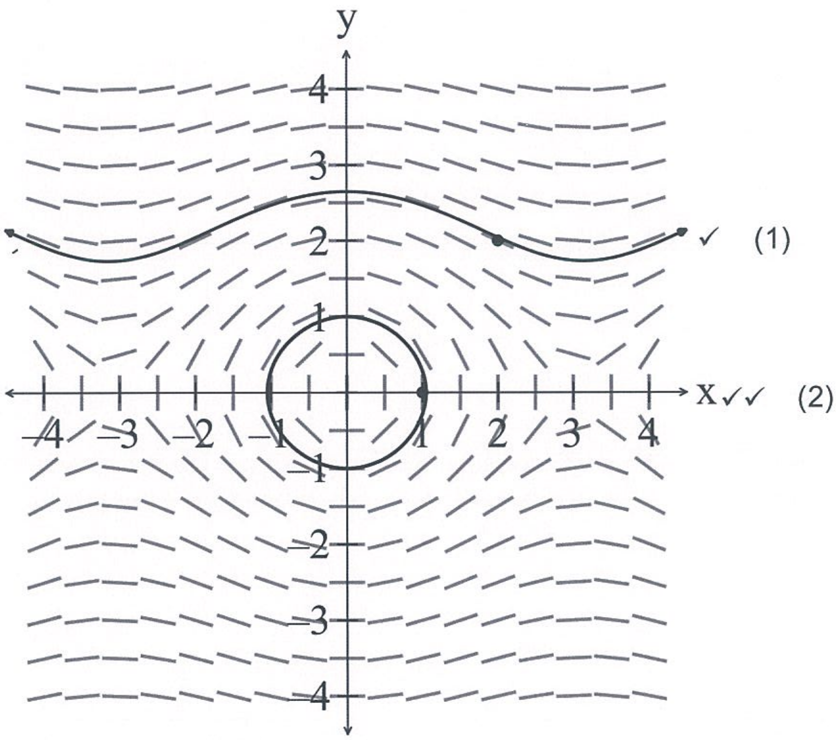
(a) Sketch the function that goes through the points

(i) (1, 0) (2)

(ii) (2, 2) (1)

on the direction field below.

(a) (i) (ii)

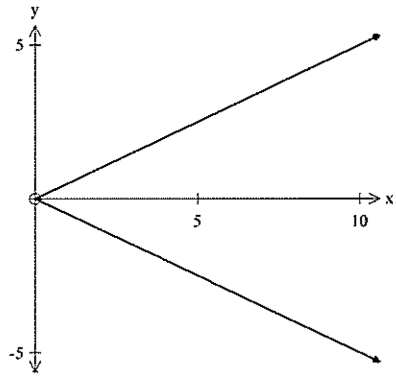


(b) What geometric conclusions can be made about the vectors **a**, **b** and **c** if $\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = 0$? Assume that each vector is not equal to the zero vector and that **b** and **c** are not parallel vectors. (2)

b and **c** can be any two non-identical vectors.
The two non-parallel vectors can be used to define a plane through a point.
The cross product forms a vector which is perpendicular to the plane containing **b** and **c**. ✓
Since the dot product equals zero, the vectors **a** and $\mathbf{b} \times \mathbf{c}$ are perpendicular.
Therefore the vector **a** is parallel to the plane containing **b** and **c**. ✓
It does NOT mean **a** is parallel to either **b** or **c** necessarily.

Question 21 **(4 marks)**

Use integration to find the volume of a cone with height 10cm and base circumference 10π cm.



✓

$$C = 2\pi r$$
$$10\pi = 2\pi r \quad ✓$$
$$r = 5 \quad ✓$$
$$y = \frac{1}{2}x$$
$$V = \int_0^{10} \pi \left(\frac{1}{2}x\right)^2 dx \quad ✓$$
$$= \frac{250}{3}\pi \ cm^2$$

End of questions

Additional working space

Question number: _________

Additional working space

Question number: __________